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Non-geometrical symmetry and separation of variables in the two-centre problem with a confinement-type potential

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Abstract

An additional spheroidal integral of motion and bound with its groups of some additional symmetry in the model quantum-mechanical problem of two centres $Z_1Z_2\omega$ with Coulomb and oscillator interactions is obtained, the group properties of its solutions are studied. For such symmetry groups of the problem we use the groups $\mathcal{P}(3) \otimes \mathcal{P}(2, 1)$, $\mathcal{P}(5, 1)$ and $\mathcal{P}(4, 2)$, among them the $\mathcal{P}(3) \otimes \mathcal{P}(2, 1)$ group possesses the smallest number of parameters. The obtained results may appear useful in the calculations of QQq-baryons and QQg-mesons energy spectra.

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1. Introduction

As a rule, when systems possessing additional (besides geometrical) symmetry are considered, two methods are used [1, 2]. The first of them consists in rewriting the Schrödinger equation and putting it in the form where such additional symmetry becomes explicit. The second one implies the construction of integrals of motion which play the role of non-geometrical symmetry group generators.

In the proposed paper, based on the example of a physically important model of confinement-type two-centred potential, we try to emphasize the deep relationship of the additional symmetry to the possibility of separation of variables in the Schrödinger equation. The knowledge of such kinds of relationships in two recent decades [3] has resulted in the intense application of the method of the separation of variables to the equations of mathematical physics and led to a series of important and far from trivial results in this field of mathematics (see, for instance, [4, 5]).

Below, the group properties of a model quantum problem of the motion of a light particle (a gluon) in the field of two heavy particles (a quark–antiquark pair) are studied. Recently, this problem has become the subject of intense study due to its relation to a wide range of problems of hadron physics: models of baryons with two heavy quarks (QQq-baryons) [6] and models of heavy hybrid mesons with open flavour (QQg-mesons) [7]. In spite of the lack of strict theoretical substantiation, the potential models give a satisfactory description of mass spectra for heavy mesons and baryons (see, e.g., [6–8] and references therein), which, according to modern views, represent bound states of quarks. While modelling the interquark interaction potential, as a rule, confinement-type potentials are used [8, 9]. One such potential is a so-called Cornell potential, containing a Coulomb-like term of single-gluon exchange and a term, responsible for the string interaction, providing the quark confinement. The confinement part of the potential is most often modelled by a spatial spherically symmetrical oscillator potential [6, 7]. Then in a non-relativistic approximation, the motion of a light quark (gluon) in the field of two heavy quarks can be described by a stationary Schrödinger equation with a model-combined potential, being the sum of the potential of two Coulomb centres and the potential of two harmonic oscillators:

$$V(r_1, r_2) = -\frac{Z_1}{r_1} - \frac{Z_2}{r_2} + \omega^2\left(r_1^2 + r_2^2\right). \qquad (1)$$

In this formula, r_1 and r_2 are the distances from the particle to the fixed force centres 1 and 2, $Z_{1,2} = \frac{2}{3}\alpha_s$, α_s is the strong interaction constant, and the phenomenological parameter ω is chosen from the condition of the best agreement of the calculated mass spectra of the quark system with the experimental data. In order to avoid ambiguities, one should mention that in our consideration, concerning not only the case of purely Coulomb interaction of the light particle with each of the centres, the notion of the force centre is preserved for the $r_{1,2} = 0$ points, where the combined potential (1) has singularities.

In the dimensionless variables, the Schrödinger equation with the model potential (1) is given by

$$\widehat{H}\Psi \equiv \left[-\frac{1}{2}\triangle - \frac{Z_1}{r_1} - \frac{Z_2}{r_2} + \omega^2\left(r_1^2 + r_2^2\right)\right]\Psi(\mathbf{r}; R) = E(R)\Psi(\mathbf{r}; R), \qquad (2)$$

where r is the distance from the particle to the midpoint of the intercentre distance R, $E(R)$ and $\Psi(\mathbf{r}; R)$ are the particle energy and wavefunction. Hereafter, the spectral problem for the Schrödinger equation (2) with the combined potential (1) is conveniently denoted by $Z_1Z_2\omega$. The sense of such notation follows from the fact that the traditional quantum-mechanical problem of two purely Coulomb centres [10] has a standard notation Z_1Z_2. Note that the Schrödinger equation for the Z_1Z_2 problem can be obtained from equation (2) by a limiting transition $\omega \to 0$.

It is known that the one-Coulomb-centre (hydrogen-like atom) problem can be solved similar to the problem of representation theory of the group $O(4)$ [1, 11] or of more general groups $O(4, 1)$ [12] and $O(4, 2)$ [13]. The wavefunctions of a hydrogen-like atom represent the special basis of a degenerate representation of the group $O(4)$ or the above mentioned more general groups. Recurrent relations between themselves connecting corresponding radial integrals $\langle r^n\rangle$ [14, 15] are the consequence of these group properties.

The group properties, eigenfunction and eigenvalue spectrum for the problem of two pure Coulomb centres have also been substantially studied in [10, 16–19]. Namely, the choice of a certain non-canonical basis in a group being a direct product of two groups of motions of three-dimensional spaces $\mathcal{P}(3) \otimes \mathcal{P}(2, 1)$, or in wider groups of motions of six-dimensional spaces $\mathcal{P}(5, 1)$ and $\mathcal{P}(4, 2)$ is known to result in the necessity to solve the problem equivalent to Z_1Z_2. The consequence of these group properties of Z_1Z_2 solutions problem is a linear algebra of two-centred integrals, obtained in [19].

Here, we show that for our case of $Z_1Z_2\omega$ problem a generally similar situation takes place. This problem can also be considered as a problem of the theory of representations of certain non-compact groups, where the function being a product of a quasiradial and a quasiangular two-centred function by $\exp(im\alpha + i\widetilde{m}\beta)$, comprises the basis of a degenerate non-canonical representation of the group being a direct product of two three-dimensional space motion groups $\mathcal{P}(3) \otimes \mathcal{P}(2, 1)$, or wider six-dimensional space motion groups $\mathcal{P}(5, 1)$, $\mathcal{P}(4, 2)$ etc.

In contrast to the one-centre problem the operator $\hat{E}$, corresponding to the energy operator of the $Z_1Z_2\omega$ problem, is not the Casimir operator of the considered groups and consequently does not commute with all generators of these groups. The operator $\widehat{E}$ and the operator $\widehat{\lambda}$, corresponding to the 'additional' operator of the separation constant λ_j commuting with $\widehat{E}$, in the two-centre problem considered here, are included into the set of mutually commuting operators, determining the non-canonical basis in the considered groups. It is connected with the fact that the Schrödinger equation (2) with two Coulomb and two oscillator potentials permits the separation of variables in the only coordinate system (in contrast to the one-Coulomb-centre problem, which enables the separation of variables in a few various coordinate systems) and also that a degeneracy of energy values (the crossing of curves at some values of internuclear distance R) exists only when out of the three not less than two quantum numbers differ, while in the one-centre problem at the given value of principal quantum number N the energy is degenerate at all possible values of quantum numbers l, m.

Also note that the considered groups $\mathcal{P}(3) \otimes \mathcal{P}(2, 1)$, $\mathcal{P}(5, 1)$, $\mathcal{P}(4, 2)$ are the groups of motions (translations and rotations) of the corresponding spaces, not the groups of rotations as in the case of the hydrogen-like atom groups.

Without a detailed consideration of all aspects of the chosen representations of the mentioned groups only note that all these representations are non-canonical representations in the group theory. For example, for a group of three-dimensional rotations the non-canonical representations were first considered in [20] in the context of the quantum theory of the asymmetric rotator.

The term 'non-canonical representation' is used in the present paper (as in [18, 19]) for faintly studied representations where not all the operators of the complete set of the observed quantities are invariants of the subgroups of the considered group.

2. Spheroidal integral of motion in the problem $Z_1Z_2\omega$

The variables in equation (2) can be separated by introducing a prolate spheroidal (elliptical) coordinate system $\{\xi\eta\alpha\}$ with the origin at the midpoint of the R segment and foci at its endpoints [10]:

$$\left.\begin{array}{ll} \xi = (r_1 + r_2)/R, & 1 \leqslant \xi < \infty, \\ \eta = (r_1 - r_2)/R, & -1 \leqslant \eta \leqslant 1, \\ \alpha = \arctan\left(\dfrac{x_2}{x_1}\right), & 0 \leqslant \alpha < 2\pi \end{array}\right\}. \tag{3}$$

Here α is the angle of rotation around the OX_3 axis; the origin of the Cartesian coordinate system $\{x_1, x_2, x_3\}$ is located at the midpoint of the segment R and the axis OX_3 is directed from the centre 1 to the centre 2.

Consider the explicit form of the differential equations resulting from the procedure of the separation of variables in equation (2) in prolate spheroidal coordinates (3):

$$\left[-\frac{2}{R^2(\xi^2-\eta^2)}\left(\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}+\frac{\xi^2-\eta^2}{(\xi^2-1)(1-\eta^2)}\frac{\partial^2}{\partial\alpha^2}\right)\right.$$
$$\left.-\frac{2Z_1}{R(\xi+\eta)}-\frac{2Z_2}{R(\xi-\eta)}+\frac{\omega^2R^2}{2}(\xi^2+\eta^2)-E\right]\Psi(\xi,\eta,\alpha;R)=0.$$

By presenting the wavefunction $\Psi(\xi,\eta,\alpha;R)$ as a product $F(\xi;R)G(\eta;R)\Phi(\alpha)$ and substituting it into (2) one obtains three ordinary differential equations linked by the separation constants λ and m:

$$\left[\frac{\mathrm{d}}{\mathrm{d}\xi}(\xi^2-1)\frac{\mathrm{d}}{\mathrm{d}\xi}+a\xi+(p^2-\gamma\xi^2)(\xi^2-1)-\frac{m^2}{(\xi^2-1)}+\lambda\right]F(\xi;R)=0, \tag{4}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}\eta}(1-\eta^2)\frac{\mathrm{d}}{\mathrm{d}\eta}+b\eta+(p^2-\gamma\eta^2)(1-\eta^2)-\frac{m^2}{(1-\eta^2)}-\lambda\right]G(\eta;R)=0, \tag{5}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}+m^2\right]\Phi(\alpha)=0. \tag{6}$$

Here, we use the notation

$$p=\frac{R}{2}\sqrt{2E'},\qquad E'=E-\frac{\omega^2R^2}{2},\qquad \gamma=\frac{\omega^2R^4}{4},$$
$$a=(Z_1+Z_2)R,\qquad b=(Z_2-Z_1)R.$$

In order to have the complete normalized wavefunction $\Psi(\mathbf{r};R)$, the functions $F(\xi;R)$ and $G(\eta;R)$ should obey the boundary conditions [10, 21]:

$$|F(1;R)|<\infty,\qquad |F(\infty;R)|<\infty \tag{7}$$

$$|G(\pm1;R)|<\infty. \tag{8}$$

The procedure for obtaining the energy terms $E(R)$ is reduced to the following steps. First two boundary problems are considered independently: (4) and (7) for the quasiradial and (5) and (8) for the quasiangular equations, $\lambda^{(\xi)}$ and $\lambda^{(\eta)}$ being considered the eigenvalues and p being left as a free parameter. Each of the eigenfunctions can be conveniently characterized by two quantum numbers n, m and the eigenvalue λ, namely: $n_\xi, m, \lambda^{(\xi)}$ for $F_{n_\xi,m}(\xi;R)$ and $n_\eta, m, \lambda^{(\eta)}$ for $G_{n_\eta,m}(\eta;R)$. The quantum numbers n_ξ, n_η are non-negative integers $0, 1, 2, \ldots,$ and coincide with the number of nodes for $F_{n_\xi,m}(\xi;R)$ and $G_{n_\eta,m}(\eta;R)$ functions on the radial $(1\leqslant\xi<\infty)$ and angular $(-1\leqslant\eta\leqslant1)$ intervals, respectively. The general theory of Sturm–Liouville-type one-dimensional boundary problems implies that the quantum numbers n_ξ, n_η, m remain constant for the continuous variation of the intercentre distance R, and the eigenvalues $\lambda^{(\xi)}_{n_\xi m}(p,a,\gamma)$ or $\lambda^{(\eta)}_{n_\eta m}(p,b,\gamma)$ are non-degenerate.

The pair of one-dimensional boundary problems for $F_{n_\xi,m}(\xi;R)$ and $G_{n_\eta,m}(\eta;R)$ is equivalent to the initial $Z_1Z_2\omega$ problem under the condition of equality of the eigenvalues $\lambda^{(\xi)}_{n_\xi m}(p,a,\gamma)=\lambda^{(\eta)}_{n_\eta m}(p,b,\gamma)$ and on account of p, a, b, γ relationship with the E, Z_1, Z_2, ω, R parameters. The eigenvalues $E_{n_\xi n_\eta m}, \lambda_{n_\xi n_\eta m}$ and eigenfunctions $\Psi_{n_\xi n_\eta m}(\mathbf{r};R)$ of the three-dimensional $Z_1Z_2\omega$ problem are enumerated by a set of quantum numbers $j=(n_\xi n_\eta m)$ which are conserved at the continuous variation of Z_1, Z_2, ω, R parameters:

$$E_j(R)=E_{n_\xi n_\eta m}(R,Z_1,Z_2,\omega), \tag{9}$$

$$\Psi_j(\mathbf{r}; R) = N_j(R) F(\xi; R) G(\eta; R) \frac{e^{im\alpha}}{\sqrt{2\pi}}. \tag{10}$$

The normalization constant $N_j(R)$ is found from the condition

$$\int_\Omega d\Omega\, \Psi_i^* \Psi_j = \delta_{ij}, \qquad d\Omega = \frac{R^3}{8}(\xi^2 - \eta^2)\, d\xi\, d\eta\, d\alpha = \frac{R^3}{8}\, d\tau\, d\alpha, \tag{11}$$

where δ_{ij} is the Kronecker symbol, $\Omega = \{\xi, \eta, \alpha \mid 1 \leqslant \xi < \infty, -1 \leqslant \eta \leqslant 1, 0 \leqslant \alpha < 2\pi\}$. Hence, the system of functions $\{\Psi_j(\mathbf{r}; R)\}$ forms a complete set of orthonormalized wavefunctions.

Now, we proceed to establish the relationship between the symmetry properties of the $Z_1 Z_2 \omega$ problem and the above separation of variables in the Schrödinger equation (2) in prolate spheroidal coordinates (3). The very fact of such separation indicates an additional (with respect to the geometrical one) symmetry of the Hamiltonian (2) causing the existence of an additional integral of motion, whose operator commutes with $\widehat{H}$ and the operator $\widehat{L}_3$, the projection of the angular moment on the intercentral axis $\mathbf{R}$. In order to reveal it, we exclude the energy parameter p^2 and the magnetic quantum number m from the above differential equation system (4)–(6). Thus, we derive the equation

$$\widehat{\lambda} \Psi_j(\mathbf{r}; R) = \lambda_j \Psi_j(\mathbf{r}; R), \tag{12}$$

where $\widehat{\lambda}$ denotes a differential operator

$$\widehat{\lambda} = \frac{1}{\xi^2 - \eta^2} \left\{ (\xi^2 - 1) \frac{\partial}{\partial \eta} (1 - \eta^2) \frac{\partial}{\partial \eta} - (1 - \eta^2) \frac{\partial}{\partial \xi} (\xi^2 - 1) \frac{\partial}{\partial \xi} \right\} + \left[\frac{1}{1 - \eta^2} - \frac{1}{\xi^2 - 1} \right] \times \frac{\partial^2}{\partial \alpha^2} - R Z_1 \frac{\xi\eta + 1}{\xi + \eta} + R Z_2 \frac{\xi\eta - 1}{\xi - \eta} + \frac{\omega^2 R^4}{4} (\xi^2 - 1)(1 - \eta^2). \tag{13}$$

The separation constant λ_j is the eigenvalue of this operator, and the solutions of equation (2) are its eigenfunctions. Since in the limit $\omega \to 0$ the model $Z_1 Z_2 \omega$ problem is reduced to the problem of two purely Coulomb centres $Z_1 Z_2$ [10], it is *a priori* obvious that the operator $\widehat{\lambda}$ should be a linear combination of the operators $\widehat{L}_3$, $\widehat{P}_3^2$ and $\widehat{H}$ (here L is the orbital moment operator and $\widehat{P}_3$ is the third component of the momentum) which in the considered limit is reduced to the operator of the separation constant for the $Z_1 Z_2$ problem [10]. To determine the weight factors and the free constant in the mentioned linear combination we compare expression (13) with the explicit form of the operators $\widehat{L}_3$, $\widehat{P}_3^2$ and $\widehat{H}$ in the prolate spheroidal coordinates (3). After simple but rather tedious calculations we finally obtain the algebraic expression for the separation constant operator in the $Z_1 Z_2 \omega$ problem:

$$\widehat{\lambda} = -\widehat{L}^2 + x_3 R \left(\frac{Z_2}{r_2} - \frac{Z_1}{r_1} \right) - \omega^2 R^2 \left(x_3^2 + \frac{R^2}{4} \right) + \frac{R^2}{4} \big(2\widehat{H} - \widehat{P}_3^2 \big). \tag{14}$$

The fact that the operator $\widehat{\lambda}$ commuting with the Hamiltonian $\widehat{H}$ and the operator $\widehat{L}_3$ of the angular moment project onto the intercentre axis $\mathbf{R}$ can easily be verified by the direct calculations of commutational relations $[\widehat{H}, \widehat{\lambda}] = [\widehat{\lambda}, \widehat{L}_3] = 0$. Thus, the operators $\widehat{H}, \widehat{L}_3, \widehat{\lambda}$ have a common complete system of eigenfunctions and can be diagonalized simultaneously. The given representation corresponds to the separation of variables in equation (2) in the prolate spheroidal coordinates (3): the general eigenfunction of the operators $\widehat{H}$, $\widehat{L}_3, \widehat{\lambda}$ is described as a product (10).

The purely geometric symmetry group of the Hamiltonian $Z_1 Z_2 \omega$ is the O_2 group containing rotations around the intercentre axis $\mathbf{R}$ and reflections in the planes containing this axis. In the symmetrical case ($Z_1 = Z_2 = Z$), the $ZZ\omega$ system possesses an additional element of geometrical symmetry—the reflection in the plane, perpendicular to the **R** vector and cutting it at its centre.

In addition to the geometrical symmetry, the $Z_1Z_2\omega$ problem possesses some higher symmetry related to the exact separation of variables in the Schrödinger equation (2) in the prolate spheroidal coordinates (3).

In the following subsections we show how, by means of the separation of variables method, the group of non-geometrical symmetry connected with the 'additional' operator of the separation constant λ_j commuting with $\widehat{E}$ can be determined for the quantum-mechanical problem $Z_1Z_2\omega$.

3. The representations of the group $\mathcal{P}(3) \otimes \mathcal{P}(2,1)$

Consider a group $\mathcal{P}(3) \otimes \mathcal{P}(2,1)$ being a direct product of two groups of motion of three-dimensional spaces $\mathcal{P}(3)$ and $\mathcal{P}(2,1)$.

We recall that the group $\mathcal{P}(3)$ (known also as the Euclidean group $E(3)$) consists of displacements (translations) and rotations (revolutions) of the Euclidean momentum space (that is conjugated to the configurational one) of coordinates y_i with a metric

$$y_i y_i = y_1^2 + y_2^2 + y_3^2, \qquad i = 1, 2, 3. \tag{15}$$

Here and below the twice repeated indices imply summation.

The $\mathcal{P}(2,1)$ group (the Euclidean group of the three-dimensional momentum space, denoted also as $E(2,1)$) consists of translations and rotations of the pseudo-Euclidean space of coordinates y_μ with a metric

$$y_\mu y_\mu = y_4^2 + y_5^2 - y_6^2, \qquad \mu = 4, 5, 6. \tag{16}$$

The infinitesimal generators of the $\mathcal{P}(3)$ group

$$x_j = -\mathrm{i}\frac{\partial}{\partial y_j}, \qquad £_{jk} = -\mathrm{i}\left(y_j\frac{\partial}{\partial y_k} - y_k\frac{\partial}{\partial y_j}\right), \qquad j, k = 1, 2, 3 \tag{17}$$

and of the $\mathcal{P}(2,1)$ group

$$\begin{aligned} &x_\mu = -\mathrm{i}\frac{\partial}{\partial y_\mu}, \qquad \mu = 4,5,6; \qquad && £_{46} = -\mathrm{i}\left(y_4\frac{\partial}{\partial y_6} + y_6\frac{\partial}{\partial y_4}\right), \\ &£_{56} = -\mathrm{i}\left(y_5\frac{\partial}{\partial y_6} + y_6\frac{\partial}{\partial y_5}\right), && £_{45} = -\mathrm{i}\left(y_4\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial y_4}\right) \end{aligned} \tag{18}$$

can easily be verified to satisfy the known structure relations:

$$\begin{aligned} &[x_i, x_j] = 0, && [x_i, £_{jk}] = \mathrm{i}(\delta_{ik}x_j - \delta_{ij}x_k), && \delta_{ij} = \begin{cases} 1, & i = j = 1,2,3 \\ 0, & i \neq j, \end{cases} \\ &[£_{12}, £_{23}] = \mathrm{i}£_{31}, && [£_{31}, £_{12}] = \mathrm{i}£_{23}, && [£_{23}, £_{31}] = \mathrm{i}£_{12}, \\ &[x_\mu, x_\nu] = 0, && [x_\sigma, £_{\mu\nu}] = \mathrm{i}(\delta_{\sigma\nu}x_\mu - \delta_{\sigma\mu}x_\nu), && \delta_{\mu\nu} = \begin{cases} 1, & \mu = \nu = 4,5 \\ -1, & \mu = \nu = 6 \\ 0, & \mu \neq \nu, \end{cases} \\ &[£_{46}, £_{56}] = \mathrm{i}£_{45}, && [£_{56}, £_{45}] = \mathrm{i}£_{46}, && [£_{45}, £_{46}] = \mathrm{i}£_{56}, \\ &[x_i, x_\mu] = 0, && [£_{ij}, £_{\mu\nu}] = 0, && [x_i, £_{\mu\nu}] = 0, \qquad [x_\mu, £_{ij}] = 0. \end{aligned} \tag{19}$$

Here, the indices i, j, k are 1, 2, 3 and μ, ν, σ are 4, 5, 6, and x_i, y_i are coordinate projections and canonically conjugate momentum. Note that in order to simplify the notation the '^' symbol over the operators is omitted since in this context no threat of ambiguity can arise.

The differential operators (17) and (18) act in the space of functions $f_j(\mathbf{y})$ which depend on the choice of the complete set of diagonal operators in $\mathcal{P}(3) \otimes \mathcal{P}(2,1)$. Here, j is the set of the eigenvalues of these operators. It is worth noting that in the chosen representation the functions $f_j(\mathbf{y})$ are scalar. In the general case, the generators (17) and (18) can possess a spin part, and $f_j(\mathbf{y})$ can be spinors, vectors, tensors, respectively.

By Fourier transformation

$$f_j(\mathbf{y}) = \int \exp(-\mathrm{i}\mathbf{x}\mathbf{y})\Psi_j(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{20}$$

we proceed to the x-representation and choose, in the $\mathcal{P}(3) \otimes \mathcal{P}(2,1)$ group, the following set of diagonal mutually commuting operators:

$$\widehat{C}_1 = x_i x_i, \qquad \widehat{C}_2 = x_i x_i \mathbf{L}^2 - x_i x_j £_{ik} £_{jk}, \tag{21}$$

$$\widehat{C}_3 = x_\mu x_\mu, \qquad \widehat{C}_4 = x_\mu x_\mu \mathbf{M}^2 - x_\mu x_\sigma £_{\nu\mu} £_{\sigma\mu}, \tag{22}$$

$$£_{12}, \qquad £_{45}, \tag{23}$$

$$\widehat{E} = -\frac{1}{2\left(x_6^2 - x_3^2\right)}\left[-\mathbf{M}^2 + 2\bar{a}x_6 - \mathbf{L}^2 + 2\bar{b}x_3 - 4\omega^2\left(x_6^4 - x_3^4\right)\right], \tag{24}$$

$$\widehat{\lambda} = \frac{\left(x_3^2 - \widehat{C}_1\right)}{\left(x_6^2 - x_3^2\right)}\left[-\mathbf{M}^2 + 2\bar{a}x_6 + 4\omega^2\left(\widehat{C}_1^2 - x_6^4\right)\right] + \frac{\left(x_6^2 - \widehat{C}_1\right)}{\left(x_6^2 - x_3^2\right)}\left[-\mathbf{L}^2 + 2\bar{b}x_3 - 4\omega^2\left(\widehat{C}_1^2 - x_3^4\right)\right]. \tag{25}$$

Here,

$$\mathbf{L}^2 = £_{12}^2 + £_{32}^2 + £_{31}^2, \qquad \mathbf{M}^2 = £_{46}^2 + £_{56}^2 - £_{45}^2,$$

ω, $\bar{a}$ and $\bar{b}$ are constants. Note that summation over the indices i, j, k is performed according to the metric (15), and over the indices μ, ν, σ according to the metric (16).

The introduced operators (21)–(25) possess important properties. The operators $\widehat{C}_1$, $\widehat{C}_2$ are the Casimir operators of the $\mathcal{P}(3)$ group and the operators $\widehat{C}_3$, $\widehat{C}_4$ are the Casimir operators of the $\mathcal{P}(2,1)$ group. One can verify by direct calculations that the operators $\widehat{C}_2$ and $\widehat{C}_4$ are equal to zero: $\widehat{C}_2 = \widehat{C}_4 = 0$. This, in turn, means that the considered representation is degenerate. Further, $£_{12}$, $£_{45}$ are the invariants of one-parametric subgroups of rotations in $\mathcal{P}(3)$ and $\mathcal{P}(2,1)$, respectively, and $\widehat{E}$, $\widehat{\lambda}$ are the non-canonical diagonal operators.

By substituting the expression for $\mathbf{L}^2 - 2\bar{b}x_3 - 4\omega^2 x_3^4$ (or $\mathbf{M}^2 - 2\bar{a}x_6 + 4\omega^2 x_6^4$) from (24) into (25) $\widehat{\lambda}$ can be given by

$$\widehat{\lambda} = -\mathbf{L}^2 + 2\left(\widehat{C}_1 - x_3^2\right)\left[\widehat{E} - 2\omega^2\left(\widehat{C}_1 + x_3^2\right)\right] + 2\bar{b}x_3, \tag{26}$$

or also by

$$\widehat{\lambda} = \mathbf{M}^2 + 2\left(\widehat{C}_1 - x_6^2\right)\left[\widehat{E} - 2\omega^2\left(\widehat{C}_1 + x_6^2\right)\right] - 2\bar{a}x_6. \tag{27}$$

Our next goal is to construct the basis of eigenvectors $\Psi_j(\mathbf{x})$ in which the complete set of operators (21)–(25) is diagonal in the $\mathcal{P}(3) \otimes \mathcal{P}(2,1)$ group. For this purpose, we introduce a new coordinate system in the x-space:

$$\begin{aligned} x_1 &= \frac{R}{2}\sqrt{1-\eta^2}\cos\alpha, & x_2 &= \frac{R}{2}\sqrt{1-\eta^2}\sin\alpha, & x_3 &= \frac{R}{2}\eta, \\ x_4 &= \frac{R}{2}\sqrt{\xi^2-1}\cos\beta, & x_5 &= \frac{R}{2}\sqrt{\xi^2-1}\sin\beta, & x_6 &= \frac{R}{2}\xi, \end{aligned} \tag{28}$$

where

$$0 \leqslant R < \infty, \qquad 1 \leqslant \xi < \infty, \qquad -1 \leqslant \eta < +1, \qquad 0 \leqslant \alpha, \beta \leqslant 2\pi. \tag{29}$$

Having omitted the intermediate calculations, we write the final expressions for the operators (17) and (18) in the new variables (28):

$$\begin{aligned}
&£_{23} = -\mathrm{i}\left(\sqrt{1-\eta^2}\sin\alpha\frac{\partial}{\partial\eta} - \eta\frac{\cos\alpha}{\sqrt{1-\eta^2}}\frac{\partial}{\partial\alpha}\right),\\
&£_{31} = -\mathrm{i}\left(-\sqrt{1-\eta^2}\cos\alpha\frac{\partial}{\partial\eta} - \eta\frac{\sin\alpha}{\sqrt{1-\eta^2}}\frac{\partial}{\partial\alpha}\right), \qquad £_{12} = -\mathrm{i}\frac{\partial}{\partial\alpha},\\
&£_{46} = -\mathrm{i}\left(\sqrt{\xi^2-1}\sin\beta\frac{\partial}{\partial\xi} + \xi\frac{\cos\beta}{\sqrt{\xi^2-1}}\frac{\partial}{\partial\beta}\right),\\
&£_{56} = -\mathrm{i}\left(\sqrt{\xi^2-1}\cos\beta\frac{\partial}{\partial\xi} - \xi\frac{\sin\beta}{\sqrt{\xi^2-1}}\frac{\partial}{\partial\beta}\right), \qquad £_{45} = -\mathrm{i}\frac{\partial}{\partial\beta}.
\end{aligned} \tag{30}$$

It is seen from these formulae that the operators $£_{12}$ and $£_{45}$, belonging to the complete set of mutually commuting operators (21)–(25), in the coordinate system (28) depend only on the variables α and β. Hence, we obtain the following relations:

$$\begin{aligned}
-\mathrm{i}\frac{\partial}{\partial\alpha}\Psi_j(\xi,\eta,R,\alpha,\beta) &= m_j\Psi_j(\xi,\eta,R,\alpha,\beta),\\
-\mathrm{i}\frac{\partial}{\partial\beta}\Psi_j(\xi,\eta,R,\alpha,\beta) &= \widetilde{m}_j\Psi_j(\xi,\eta,R,\alpha,\beta),
\end{aligned} \tag{31}$$

where $m_j, \widetilde{m}_j$ are the eigenvalues of the $£_{12}, £_{45}$, operators, respectively. The common solution of equations (31) can now be given in the multiplicative form

$$\Psi_j(\xi,\eta,R,\alpha,\beta) = \varphi(\xi,\eta,R)\,\mathrm{e}^{\mathrm{i}m_j\alpha+\mathrm{i}\widetilde{m}_j\beta}. \tag{32}$$

The rest of the operators from the complete set (21)–(25) in the coordinate system (28) taking account of (31) are given by

$$\widehat{C}_1 = \frac{R^2}{4}, \qquad \widehat{C}_2 = 0, \qquad \widehat{C}_3 = -\frac{R^2}{4}, \qquad \widehat{C}_4 = 0, \tag{33}$$

$$\begin{aligned}
\widehat{E} = &-\frac{2}{R^2(\xi^2-\eta^2)}\left[\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi} + a\xi - \frac{\omega^2R^4}{4}\xi^4 - \frac{\widetilde{m}_j^2}{\xi^2-1}\right]\\
&-\frac{2}{R^2(\xi^2-\eta^2)}\left[\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta} + b\eta + \frac{\omega^2R^4}{4}\eta^4 - \frac{m_j^2}{1-\eta^2}\right],
\end{aligned} \tag{34}$$

$$\begin{aligned}
\widehat{\lambda} = &-\frac{(1-\eta^2)}{(\xi^2-\eta^2)}\left[\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi} + a\xi + \frac{\omega^2R^4}{4}(1-\xi^4) - \frac{\widetilde{m}_j^2}{\xi^2-1}\right]\\
&+\frac{(\xi^2-1)}{(\xi^2-\eta^2)}\left[\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta} + b\eta - \frac{\omega^2R^4}{4}(1-\eta^4) - \frac{m_j^2}{1-\eta^2}\right],
\end{aligned} \tag{35}$$

note that $a = \bar{a}R$, $b = \bar{b}R$.

Though in order to solve the question concerning the eigenfunctions of the complete set of operators (21)–(25) one can use their explicit form (33)–(35), we give the expressions for the operators (26) and (27) in the new coordinates as well, since they will also be used for another purpose:

$$\widehat{\lambda} = \left[\frac{\partial}{\partial \eta}(1-\eta^2)\frac{\partial}{\partial \eta} + b\eta + (1-\eta^2)\frac{R^2\widehat{E}}{2} - \frac{\omega^2 R^4}{4}(1-\eta^4) - \frac{m_j^2}{1-\eta^2} \right], \tag{36}$$

$$\widehat{\lambda} = -\left[\frac{\partial}{\partial \xi}(\xi^2-1)\frac{\partial}{\partial \xi} + a\xi + (\xi^2-1)\frac{R^2\widehat{E}}{2} + \frac{\omega^2 R^4}{4}(1-\xi^4) - \frac{\widetilde{m}_j^2}{\xi^2-1} \right]. \tag{37}$$

Now we show how, using the separation of variables method, one can find $\Psi_j(\xi, \eta, R, \alpha, \beta)$ functions which are common eigenfunctions of the operators (31), (33)–(35) and (36), (37). The application of this method is based on the properties of the $\widehat{\lambda}$ operator, expressed by equations (36) and (37). We choose the basis of eigenvectors Ψ_j where the operator $\widehat{\lambda}$ is diagonal:

$$\widehat{\lambda}\Psi_j = \lambda_j \Psi_j \tag{38}$$

and represent Ψ_j in the form of a product

$$\Psi_j \equiv \Psi_j(\xi, \eta, R, \alpha, \beta) = N_j(R) F_j(\xi; R) G_j(\eta; R) \frac{\exp(\mathrm{i}m_j\alpha + \mathrm{i}\widetilde{m}_j\beta)}{\sqrt{2\pi}}, \tag{39}$$

where λ_j are the eigenvalues of the operator $\widehat{\lambda}$ and $N_j(R)$ is a normalization factor. After the separation of variables in (38) a pair of ordinary differential equations for the unknown functions $F_j(\xi; R)$ and $G_j(\eta; R)$ is obtained:

$$\left[\frac{\partial}{\partial \xi}(\xi^2-1)\frac{\partial}{\partial \xi} + a\xi + \frac{R^2 E_j}{2}(\xi^2-1) + \frac{\omega^2 R^4}{4}(1-\xi^4) + \lambda_j - \frac{\widetilde{m}_j^2}{\xi^2-1} \right] F_j(\xi; R) = 0, \tag{40}$$

$$\left[\frac{\partial}{\partial \eta}(1-\eta^2)\frac{\partial}{\partial \eta} + b\eta + \frac{R^2 E_j}{2}(1-\eta^2) - \frac{\omega^2 R^4}{4}(1-\eta^4) - \lambda_j - \frac{m_j^2}{1-\eta^2} \right] G_j(\eta; R) = 0. \tag{41}$$

Here, E_j are the eigenvalues of the operator $\widehat{E}$. Since the operator $\widehat{\lambda}$ commutes with all the operators (21)–(24), the eigenfunctions (39) of the operator $\widehat{\lambda}$ are also the eigenfunctions of the operators (21)–(24) in the coordinate system (28).

The invariance of the Hamiltonian of the $Z_1 Z_2 \omega$ problem with respect to the $\mathcal{P}(3) \otimes \mathcal{P}(2, 1)$ group is now obvious. Indeed, at $m_j = \widetilde{m}_j = m$ the system of equations (40) and (41) coincides with the system (4)–(6). Hence, at given $a, b, \omega, R, m_j, \widetilde{m}_j$ the determination of the eigenvalues $E_j = E_j(R), \lambda_j = \lambda_j(R)$ and, limited in the corresponding ranges (29), eigenfunctions $F_j(\xi; R), G_j(\eta; R)$ of the complete set of the commuting operators (31), (33)–(35) is reduced at $m_j = \widetilde{m}_j = m$ to the solution of the problem, completely equivalent to the quantum-mechanical problem $Z_1 Z_2 \omega$. In this case, the common eigenfunctions (39) of the complete set (31) and (33)–(35), which comprise the basis of the degenerate non-canonical representation of $\mathcal{P}(3) \otimes \mathcal{P}(2, 1)$ group, coincide within the normalization factor with the two-centred functions (10) multiplied by $\exp(\mathrm{i}m\beta)$. Expressions (34) and (35) for the operators $\widehat{E}$ and $\widehat{\lambda}$ coincide at $m_j = \widetilde{m}_j = m$ with the expressions for the operators of energy $\widehat{H}$ and separation constant $\widehat{\lambda}$ (see (13)) in the $Z_1 Z_2 \omega$ problem in the prolate spheroidal coordinate system (3). The variable R, being used to express the Casimir operators of the $\mathcal{P}(3) \otimes \mathcal{P}(2, 1)$ group, in the $Z_1 Z_2 \omega$ problem is equal to the intercentre distance.

The operators (31) and (33)–(35) are Hermitian in the scalar product

$$\langle \Psi_i \mid \Psi_j \rangle = \int_{\overline{\Omega}} \Psi_i^* \Psi_j \, \mathrm{d}\overline{\Omega}, \tag{42}$$

where $\overline{\Omega}$ corresponds to the range (29) and the volume element $d\overline{\Omega} = \xi\, d\xi\, d\eta\, d\alpha\, d\beta$. The relation between the $d\overline{\Omega}$ from equation (42) and volume element $d\Omega$ from equation (11) as well as the link between coordinate systems (3) and (28) are given in [22]. Thus, the representation, corresponding to the set (33)–(35) and (36), (37), is unitary.

One of the possible consequences of the above group interpretation of the solutions of the $Z_1Z_2\omega$ problem consists in the calculation of the matrix elements of generators (30) in the non-canonical basis (39) being reduced to the calculation of the two-centred integrals over the variable ξ and the similar integrals over the variable η. This circumstance is the base for the deduction (without the use of the explicit form of the solutions of the system of equations (4)–(6)) of a specific linear algebra of two-centred integrals. It consists of a sum of two independent subalgebras: one for the radial integrals containing polynomials over ξ, $\sqrt{\xi^2-1}$ and $\frac{\partial}{\partial\xi}$, and the other for the angular integrals containing polynomials over η, $\sqrt{1-\eta^2}$ and $\frac{\partial}{\partial\eta}$. But in the specific quantum-mechanical calculations of the energies and wavefunctions of various states of three-quark systems the calculations of two-centred integrals, containing the derivatives over R, are required. The standard way, resulting in the construction of the algebra of such kinds of integrals, consists in the extension of the group $\mathcal{P}(3) \otimes \mathcal{P}(2,1)$ to the ordinary one $\mathcal{P}(5,1)$, being realized by the motions of the six-dimensional coordinate space y_ν with the metric

$$y_\nu y_\nu = y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 - y_6^2. \tag{43}$$

Having complemented the set of generators (17) and (18) by nine more generators

$$\begin{aligned} &£_{j4} = -\mathrm{i}\left(y_j\frac{\partial}{\partial y_4} - y_4\frac{\partial}{\partial y_j}\right), \qquad j = 1, 2, 3, \\ &£_{j5} = -\mathrm{i}\left(y_j\frac{\partial}{\partial y_5} - y_5\frac{\partial}{\partial y_j}\right), \qquad £_{j6} = -\mathrm{i}\left(y_j\frac{\partial}{\partial y_6} + y_6\frac{\partial}{\partial y_j}\right), \end{aligned} \tag{44}$$

we proceed to the x-representation and choose the set of diagonal commuting operators corresponding to the set (21)–(25). Additional diagonal operators, arising in the group $\mathcal{P}(5,1)$ due to the degeneracy of the chosen representation, do not result in any new relations. In the coordinate system (28), we obtain the same equations (40) and (41) which are reduced to the problem (4)–(6) and whose solutions in the case of the group $\mathcal{P}(5,1)$ will be realized on the cone

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_6^2 = 0. \tag{45}$$

In this case, the generators (44) in the coordinate representation in the coordinate system (28) are given by

$$\begin{aligned} £_{14} = -\mathrm{i}\sqrt{(\xi^2-1)(1-\eta^2)}\Bigg[&\cos\alpha\cos\beta\left(\xi\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta}\right) - \frac{\cos\alpha\sin\beta}{(\xi^2-1)}\frac{\partial}{\partial\beta} \\ &+ \frac{\cos\beta\sin\alpha}{(1-\eta^2)}\frac{\partial}{\partial\alpha} - R\cos\alpha\cos\beta\frac{\partial}{\partial R}\Bigg], \\ £_{24} = -\mathrm{i}\sqrt{(\xi^2-1)(1-\eta^2)}\Bigg[&\sin\alpha\cos\beta\left(\xi\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta}\right) - \frac{\sin\alpha\sin\beta}{(\xi^2-1)}\frac{\partial}{\partial\beta} \\ &- \frac{\cos\beta\cos\alpha}{(1-\eta^2)}\frac{\partial}{\partial\alpha} - R\sin\alpha\cos\beta\frac{\partial}{\partial R}\Bigg], \\ £_{34} = \mathrm{i}\sqrt{\xi^2-1}\Bigg[&\cos\beta\left(\xi\eta\frac{\partial}{\partial\xi} - (1-\eta^2)\frac{\partial}{\partial\eta}\right) - \frac{\eta\sin\beta}{(\xi^2-1)}\frac{\partial}{\partial\beta} - R\eta\cos\beta\frac{\partial}{\partial R} \\ &+ \frac{\sin\alpha\sin\beta}{(1-\eta^2)}\frac{\partial}{\partial\alpha} - R\cos\alpha\sin\beta\frac{\partial}{\partial R}\Bigg], \end{aligned}$$

$$\pounds_{25} = -\mathrm{i}\sqrt{(\xi^2-1)(1-\eta^2)}\left[\sin\alpha\sin\beta\left(\xi\frac{\partial}{\partial\xi}+\eta\frac{\partial}{\partial\eta}\right)+\frac{\sin\alpha\cos\beta}{(\xi^2-1)}\frac{\partial}{\partial\beta}\right.$$
$$\left.-\frac{\cos\alpha\sin\beta}{(1-\eta^2)}\frac{\partial}{\partial\alpha}-R\sin\alpha\sin\beta\frac{\partial}{\partial R}\right],$$
$$\pounds_{35} = -\mathrm{i}\sqrt{\xi^2-1}\left[\sin\beta\left(\xi\eta\frac{\partial}{\partial\xi}-(1-\eta^2)\frac{\partial}{\partial\eta}\right)+\frac{\eta\cos\beta}{(\xi^2-1)}\frac{\partial}{\partial\beta}-R\eta\sin\beta\frac{\partial}{\partial R}\right],$$
$$\pounds_{16} = -\mathrm{i}\sqrt{1-\eta^2}\left[\cos\alpha\left(-(\xi^2-1)\frac{\partial}{\partial\xi}-\xi\eta\frac{\partial}{\partial\eta}\right)-\frac{\xi\sin\alpha}{(1-\eta^2)}\frac{\partial}{\partial\alpha}+R\xi\cos\alpha\frac{\partial}{\partial R}\right],$$
$$\pounds_{26} = -\mathrm{i}\sqrt{1-\eta^2}\left[\sin\alpha\left(-(\xi^2-1)\frac{\partial}{\partial\xi}-\xi\eta\frac{\partial}{\partial\eta}\right)+\frac{\xi\cos\alpha}{(1-\eta^2)}\frac{\partial}{\partial\alpha}+R\xi\sin\alpha\frac{\partial}{\partial R}\right],$$
$$\pounds_{36} = -\mathrm{i}\left[-\eta(\xi^2-1)\frac{\partial}{\partial\xi}+\xi(1-\eta^2)\frac{\partial}{\partial\eta}+\xi\eta R\frac{\partial}{\partial R}\right]. \tag{46}$$

Finally, consider the basis in the group $\mathcal{P}(4,2)$—the group of motions of the six-dimensional coordinate space y_μ with a metric

$$y_\mu y_\mu = y_1^2+y_2^2+y_3^2-y_4^2-y_5^2+y_6^2. \tag{47}$$

We introduce the infinitesimal generators of this group

$$x_j = -\mathrm{i}\frac{\partial}{\partial y_j}, \qquad L_{jk} = -\mathrm{i}\left(y_j\frac{\partial}{\partial y_k}-y_k\frac{\partial}{\partial y_j}\right), \qquad j,k=1,2,3,6,$$
$$L_{\mu k} = -\mathrm{i}\left(y_\mu\frac{\partial}{\partial y_k}+y_k\frac{\partial}{\partial y_\mu}\right), \qquad L_{45} = -\mathrm{i}\left(y_4\frac{\partial}{\partial y_5}-y_5\frac{\partial}{\partial y_4}\right), \qquad \mu=4,5 \tag{48}$$

and proceed in the x-representation to a new coordinate system

$$x_1=\frac{R}{\sqrt2}\sqrt{1-\eta^2}\cos\alpha\cos\theta, \qquad x_2=\frac{R}{\sqrt2}\sqrt{1-\eta^2}\sin\alpha\cos\theta, \qquad x_3=\frac{R}{\sqrt2}\eta\cos\theta,$$
$$x_4=\frac{R}{\sqrt2}\sqrt{\xi^2-1}\cos\beta\sin\theta, \qquad x_5=\frac{R}{\sqrt2}\sqrt{\xi^2-1}\sin\beta\sin\theta, \qquad x_6=\frac{R}{\sqrt2}\xi\sin\theta, \tag{49}$$

where α,β run from 0 to π, and θ from 0 to $\frac{\pi}{2}$. By calculating the expressions for the generators (48) in the x-representation in the new coordinates (49), we finally obtain that L_{jk} $(j,k=1,2,3)$, L_{56}, L_{46}, L_{45} have the same form as $\pounds_{jk}$ $(j,k=1,2,3)$, $\pounds_{56}$, $\pounds_{46}$, $\pounds_{45}$ in the $\mathcal{P}(3)\otimes\mathcal{P}(2,1)$ group in the coordinate system (28). Now following the above scheme of constructing the complete set of commuting operators of equations (21)–(25) type, we obtain at $\cos\theta=\sin\theta=\frac{1}{\sqrt2}$, $\frac{\partial}{\partial\theta}=0$ a problem, completely equivalent to the $Z_1Z_2\omega$ problem.

A similar consideration of wider groups, e.g., conformal groups of six-dimensional spaces (43) and (47) or a group being a direct product of two conformal groups of spaces (15) and (16), results in the choice of the corresponding set of commuting operators (of equations (21)–(25) type) to a problem, equivalent to the $Z_1Z_2\omega$ problem. The calculation of the matrix elements of the generators of these groups is reduced to the calculation of two-centred integrals, some of which contain the first and the second derivatives over R.

4. Conclusions and final remarks

Summarizing the results of this work, we focus on its most important points. By means of the separation of variables method an additional spheroidal integral of motion $\widehat{\lambda}$ is constructed, whose eigenvalues are the separation constant in the model quantum-mechanical $Z_1Z_2\omega$ problem. This has enabled some additional symmetry groups of this problem to be determined and the group properties of its solutions to be studied. $\mathcal{P}(3)\otimes\mathcal{P}(2,1)$, $\mathcal{P}(5,1)$ and $\mathcal{P}(4,2)$ groups are considered as such groups, among them $\mathcal{P}(3)\otimes\mathcal{P}(2,1)$ possessing the smallest number of parameters.

While searching for the eigenfunctions of the complete set of mutually commuting operators in the $\mathcal{P}(3)\otimes\mathcal{P}(2,1)$ group in the case of degenerate unitary representations of this group a problem is shown to occur, quite equivalent to the quantum-mechanical $Z_1Z_2\omega$ problem. In this case, the energy operator $\widehat{E}$ is not the Casimir operator of this group and, accordingly, does not commute with all generators of this group. The operator $\widehat{E}$ and the operator $\widehat{\lambda}$, corresponding to the 'additional' operator of the separation constant λ_j, commuting with $\widehat{E}$, are included into the set of the diagonal operators, determining the non-canonical basis in the considered group. The space of the chosen representation of this group covers the whole spectrum of the energy values for the two-centre $Z_1Z_2\omega$ problem.

The developed group treatment of the model $Z_1Z_2\omega$ problem is related to the group treatment of the traditional quantum-mechanical problem of two Coulomb centres Z_1Z_2 [10, 16–19]. But its consequence is a richer linear algebra of two-centred integrals, which contains the corresponding linear algebra of the Z_1Z_2 problem as a partial case (i.e., at $\omega = 0$). A separate publication will be devoted to the construction of such an algebra while here we only represent a relation obtained using its basis

$$\frac{\partial}{\partial R}\left(-\frac{R^2E_i}{2}\right) = -\frac{R}{2}V_{ii}, \tag{50}$$

$$R(E_i - E_j)\int \mathrm{d}\Omega\,\Psi_i^*\frac{\partial}{\partial R}\Psi_j = -V_{ij}, \qquad i\neq j, \qquad m = m', \tag{51}$$

where

$$V_{ij} = \int \mathrm{d}\Omega\,\Psi_i^*\left(-\frac{Z_1}{r_1} - \frac{Z_2}{r_2} + \omega^2\big(r_1^2 + r_2^2\big)\right)\Psi_j.$$

Expression (50) generalizes the Helman–Feynman theorem [23] and go over into it at $\omega = 0$.

The presence of both mentioned algebras enables and essentially simplifies the quantum-mechanical calculations of matrix elements and effective potentials in the three-body problem with Coulomb and oscillatory interactions [10]. In particular, the obtained results may appear useful in the calculations of the energy spectra of QQq-baryons and QQg-mesons. Also note that the model $Z_1Z_2\omega$ problem can for certain conditions be treated as a step to the solution of a relativized Schrödinger equation [24] with a two-centred confinement-type potential (1).

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